Unit 4

Application of Boolean Algebra / Minterm and Maxterm Expansion

Outline

- Conversion of sentences to Boolean equations
- Truth table-based logic design
- Minterm and maxterm expansions
- Incompletely specified functions
- Binary adders and subtracters
- Speeding up integer additions
- Binary multiplication

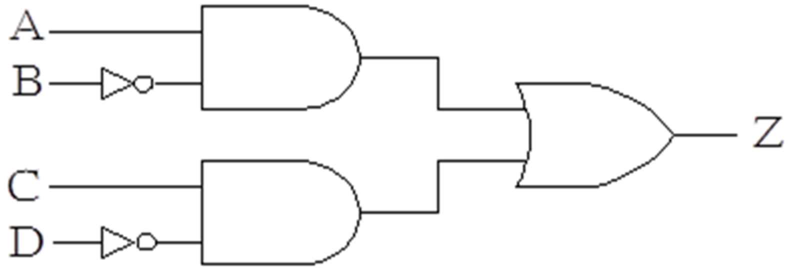
Conversion of Sentences to Boolean Equations

$$Ex: \underbrace{The\ alarm\ will\ ring}_{Z} \quad iff \quad \underbrace{the\ power\ of\ alarm\ is\ on}_{A} \quad and$$

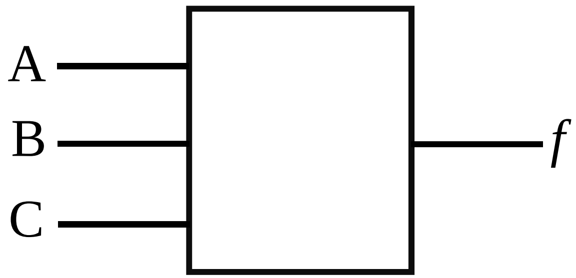
$$\underbrace{the\ door\ is\ not\ closed}_{B'} \quad or \quad \underbrace{it\ is\ after\ 6\ p.m.}_{C} \quad and$$

$$\underbrace{the\ window\ is\ not\ closed}_{D'}$$

$$Z = AB' + CD'$$



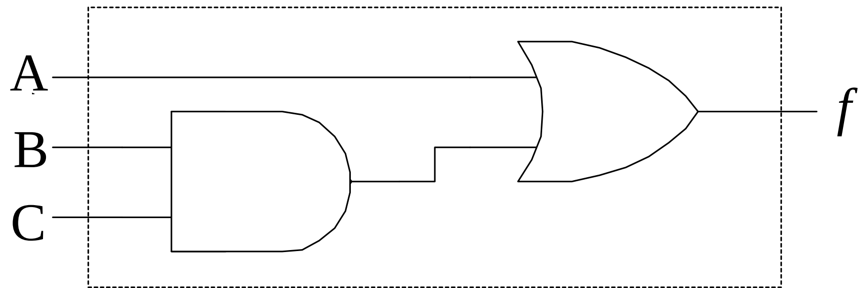
Truth Table-based Logic Design (1/2)



A	B	C	f	f'
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

① By “1's” function

$$f = A'BC + AB'C' + AB'C + ABC' + ABC$$
$$= A'BC + AB' + AB$$
$$= A'BC + A$$
$$= A + BC$$



Truth Table-based Logic Design (2/2)

② By "0's" function

$$f=(A+B+C)(A+B+C')(A+B'+C)$$
$$=(A+B)(A+B'+C)=A+B(B'+C)$$
$$=A+BC$$

f is "0" ,

$if\ \ A=B=C=0$

$A=B=0\,,C=1$

$A=0=C\,,B=1$

A	B	C	f	$f\,'$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

③ By $f\,'$

$$f\,'=A'B'C'+A'B'C+A'BC'$$
$$\Downarrow$$
$$f=(A+B+C)(A+B+C')(A+B'+C)$$

Minterm & Maxterm Expansions (1/10)

Variables $: X, Y, Z \cdots A, B, C \cdots$

Literals $: X, X', Y, Y' \cdots\cdots$

Example

$F = ABC' + A'B' + BC'$ 3 variables 7 literals

Row No.	A	B	C	Mintems			Maxterms			f	f'
0	0	0	0	A'B'C'	=	m_0	A+B+C	=	M_0	0	1
1	0	0	1	A'B'C	=	m_1	A+B+C'	=	M_1	0	1
2	0	1	0	A'BC'	=	m_2	A+B'+C	=	M_2	0	1
3	0	1	1	A'BC	=	m_3	A+B'+C'	=	M_3	1	0
4	1	0	0	AB'C'	=	m_4	A'+B+C	=	M_4	1	0
5	1	0	1	AB'C	=	m_5	A'+B+C'	=	M_5	1	0
6	1	1	0	ABC'	=	m_6	A'+B'+C	=	M_6	1	0
7	1	1	1	ABC	=	m_7	A'+B'+C'	=	M_7	1	0

Minterm & Maxterm Expansions (2/10)

$$m_i' = Mi$$

$$\Rightarrow \quad f = A'BC + AB'C' + AB'C + ABC' + ABC$$
$$= m_3 + m_4 + m_5 + m_6 + m_7$$
$$or \quad f(A, B, C) = \sum m(3, 4, 5, 6, 7)$$
$$\text{min term} = 1 \quad \Rightarrow \quad f = 1$$

$$f = (A+B+C)(A+B+C')(A+B'+C) = M_0 M_1 M_2$$
$$f(A,B,C) = \prod M(0,1,2)$$
$$\text{maxterm} = 0 \quad \Rightarrow \quad f = 0$$

Minterm & Maxterm Expansions (3/10)

$$f(A,B,C)=\sum m(3,4,5,6,7) \quad \Rightarrow \quad f'(A,B,C)=m_0+m_1+m_2$$

$$(f')'=f=(m_0+m_1+m_2)'=m_0'\cdot m_1'\cdot m_2' = M_0M_1M_2$$

$$=\prod M(0,1,2)$$

$$f' = (m_3+m_4+m_5+m_6+m_7)'=m_3'\cdot m_4'\cdot m_5'\cdot m_6'\cdot m_7'$$

$$=M_3M_4M_5M_6M_7=\prod M(3,4,5,6,7)$$

Minterm & Maxterm Expansions (4/10)

Another Example

A	B	C	D	g	g'	
0	0	0	0	0	1	m_0
0	0	0	1	0	1	m_1
0	0	1	0	1	0	m_2
0	0	1	1	1	0	m_3
0	1	0	0	1	0	m_4
0	1	0	1	0	1	m_5
0	1	1	0	1	0	m_6
0	1	1	1	1	0	m_7
1	0	0	0	0	1	m_8
1	0	0	1	0	1	m_9
1	0	1	0	1	0	m_{10}
1	0	1	1	1	0	m_{11}
1	1	0	0	0	1	m_{12}
1	1	0	1	0	1	m_{13}
1	1	1	0	1	0	m_{14}
1	1	1	1	0	1	m_{15}

$$g = \sum m(2, 3, 4, 6, 7, 10, 11, 14)$$

$$or = \prod M(0, 1, 5, 8, 9, 12, 13, 15)$$

$$g' = \sum m(0, 1, 5, 8, 9, 12, 13, 15)$$

$$g = [\sum m(0, 1, 5, 8, 9, 12, 13, 15)]'$$

$$= \prod m'(0, 1, 5, 8, 9, 12, 13, 15)$$

$$= \prod M(0, 1, 5, 8, 9, 12, 13, 15)$$

Minterm & Maxterm Expansions (5/10)

$Ex1: Find\ "Minterm\ Expansion"\ of\ f = a'(b'+d) + acd'$

$$
\begin{aligned}
f &= a'b' + a'd + acd' \\
&= a'b'(c+c')(d+d') + a'd(b+b')(c+c') + acd'(b+b') \\
&= a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'b'c'd + a'b'cd \\
&\quad + a'bc'd + a'bcd + abcd' + ab'cd' \\
&= 0000 + 0001 + 0010 + 0011 + 0101 + 0111 + 1110 + 1010 \\
&= \sum m(0,1,2,3,5,7,10,14) \\
&Find\ Maxterm\ Expansion\quad f = \prod M(4,6,8,9,11,12,13,15)
\end{aligned}
$$

$Ex2: Find\ "Maxterm\ Expansion"\ of\ f = a + \bar{b}c$

Minterm & Maxterm Expansions (6/10)

$Ex3: Prove\ a'c + b'c' + ab = a'b' + bc + ac'$

$$LHS = a'c\,(b + b') + (a + a')\,b'c' + ab\,(c + c')$$
$$= m_3 + m_1 + m_4 + m_0 + m_7 + m_6$$
$$= \sum m(0, 1, 3, 4, 6, 7)$$

$$RHS = a'b'(c + c') + (a + a')bc + ac'(b + b')$$
$$= m_1 + m_0 + m_7 + m_3 + m_6 + m_4$$
$$= \sum m(0, 1, 3, 4, 6, 7)$$

Minterm & Maxterm Expansions (7/10)

A	B	C	F
0	0	0	a_0
0	0	1	a_1
0	1	0	a_2
0	1	1	a_3
1	0	0	a_4
1	0	1	a_5
1	1	0	a_6
1	1	1	a_7

$$\mathrm{F} = a_0 m_0 + a_1 m_1 + a_2 m_2 + \ldots + a_7 m_7 = \sum_{i=0}^{7} a_i m_i \cdots\cdots (\mathrm{a})$$

If i_{th} term exists $\Rightarrow$ $a_i = 1$

OR

$$F = (a_0 + M_0)(a_1 + M_1)(a_2 + M_2)\ldots\ldots(a_7 + M_7)$$

$$= \prod_{i=0}^{7} (a_i + M_i) \cdots\cdots (b)$$

If i_{th} term does not exist $\Rightarrow$ $a_i = 1$

$\because$ $(1 + M_i) = 1$

Minterm & Maxterm Expansions (8/10)

A	B	C	F		
0	0	0	0	m_0	M_0
0	0	1	1	m_1	M_1
0	1	0	1	m_2	M_2
0	1	1	0	m_3	M_3
1	0	0	0	m_4	M_4
1	0	1	0	m_5	M_5
1	1	0	1	m_6	M_6
1	1	1	1	m_7	M_7

$$
\begin{aligned}
F &= 1 \cdot m_1 + 1 \cdot m_2 + 1 \cdot m_6 + 1 \cdot m_7 \\
&= (0 + M_0)(0 + M_3)(0 + M_4)(0 + M_5)
\end{aligned}
$$

Minterm & Maxterm Expansions (9/10)

$$\begin{aligned}
&From \quad (b) \\
&F' = [\prod (a_i + M_i)]' = \sum (a_i + M_i)' \\
&\quad = \sum (a_i' \cdot M_i') \quad = \sum (a_i' \cdot m_i)
\end{aligned}$$

$$\begin{aligned}
&From \quad (a) \\
&F' = [\sum (a_i \cdot m_i)]' \quad = \prod (a_i' + m_i') \\
&\quad = \prod (a_i' + M_i)
\end{aligned}$$

m_i not in F, will be in F'

M_i not in F, will be in F'

Minterm & Maxterm Expansions (10/10)

$$\Rightarrow \text{ General Form : F} = \sum_{i=0}^{2^n-1} a_i m_i = \prod_{i=0}^{2^n-1} (a_i + M_i)$$

$$\text{F}' = \sum_{i=0}^{2^n-1} a_i ' m_i = \prod_{i=0}^{2^n-1} (a_i ' + M_i)$$

$$property : \boxed{m_i m_j = 0 \quad if \quad \text{i} \neq \text{j}} \qquad \text{Ex:(ABCD)(AB}\overline{\text{C}}\text{D)} = 0$$

$$\text{So: } \text{f}_1 = \sum_{i=0}^{2^n-1} a_i m_i \qquad \text{f}_2 = \sum_{j=0}^{2^n-1} b_j m_j$$

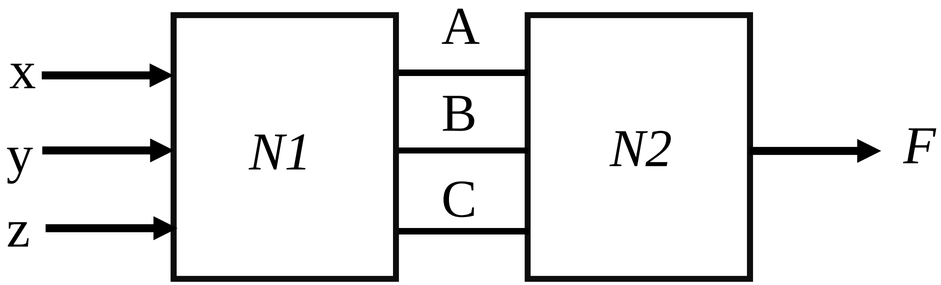
$$\text{f}_1\text{f}_2 = (\sum_{i=0}^{2^n-1} a_i m_i)(\sum_{j=0}^{2^n-1} b_j m_j) = \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} a_i b_j m_i m_j$$

$$= \Sigma a_i b_i m_i \quad (i \neq j \quad \text{terms} = 0)$$

$$\text{Ex: } \quad \text{f}_1 = \sum m(0,2,3,5,9,11) \quad \text{f}_2 = \sum m(0,3,9,11,13,14)$$

$$\text{f}_1\text{f}_2 = \sum m(0,3,9,11)$$

Incompletely Specified Functions (1/2)



N_1

x	y	z	A	B	C
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	0	1
1	1	1	1	1	1

Incompletely Specified Functions (2/2)

Ⅰ. $X, X = 0, 0$

$$F = A'B'C' + A'BC + ABC$$
$$= A'B'C' + BC$$

Ⅱ. $X, X = 1, 0$

$$F = A'B'C' + A'B'C + A'BC + ABC$$
$$= A'B' + BC$$

Ⅲ. $X, X = 1, 1$

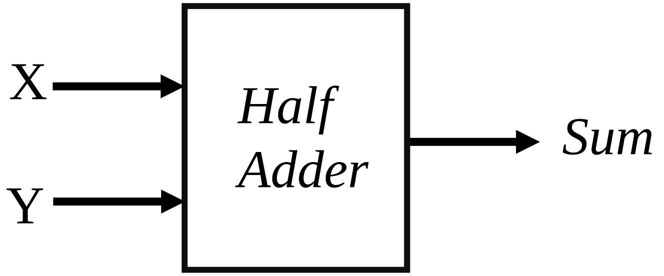
$$F = A'B'C' + A'B'C + A'BC + ABC' + ABC$$
$$= A'B' + BC + AB$$

	A	B	C	F	
	0	0	0	1	
	0	0	1	$\boxed{X}$	← don' t care
	0	1	0	0	
N_2	0	1	1	1	
	1	0	0	0	
	1	0	1	0	
	1	1	0	$\boxed{X}$	
	1	1	1	1	

$$F = \sum m(0,3,7) + \sum d(1,6)$$
$$or = \prod M(2,4,5) \cdot \prod D(1,6)$$

Binary Adders & Subtracters (1/9)

- **Half Adder**:

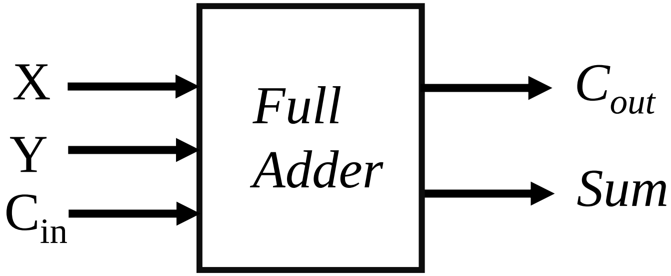


X	Y	Sum
0	0	0
0	1	1
1	0	1
1	1	0

$$Sum = X'Y + XY'$$

Binary Adders & Subtracters (2/9)

- **Full Adder**:



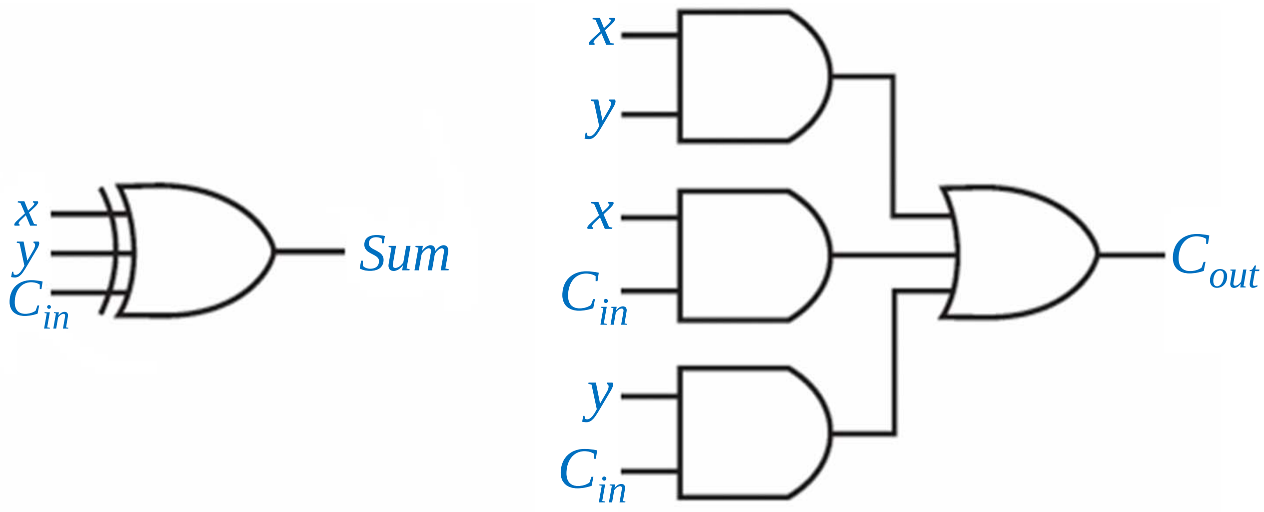
X	Y	C_{in}	C_{out}	Sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

- The logic equation for the full adder:

$$
\begin{aligned}
Sum &= X'Y'C_{in} + X'YC'_{in} + XY'C'_{in} + XYC_{in} \\
&= X'(Y'C_{in} + YC'_{in}) + X(Y'C'_{in} + YC_{in}) \\
&= X'(Y \oplus C_{in}) + X(Y \oplus C_{in})' = X \oplus Y \oplus C_{in} \\
C_{out} &= X'YC_{in} + XY'C_{in} + XYC'_{in} + XYC_{in} \\
&= (X'YC_{in} + XYC_{in}) + (XY'C_{in} + XYC_{in}) + (XYC'_{in} + XYC_{in}) \\
&= YC_{in} + XC_{in} + XY
\end{aligned}
$$

Binary Adders & Subtracters (4/9)

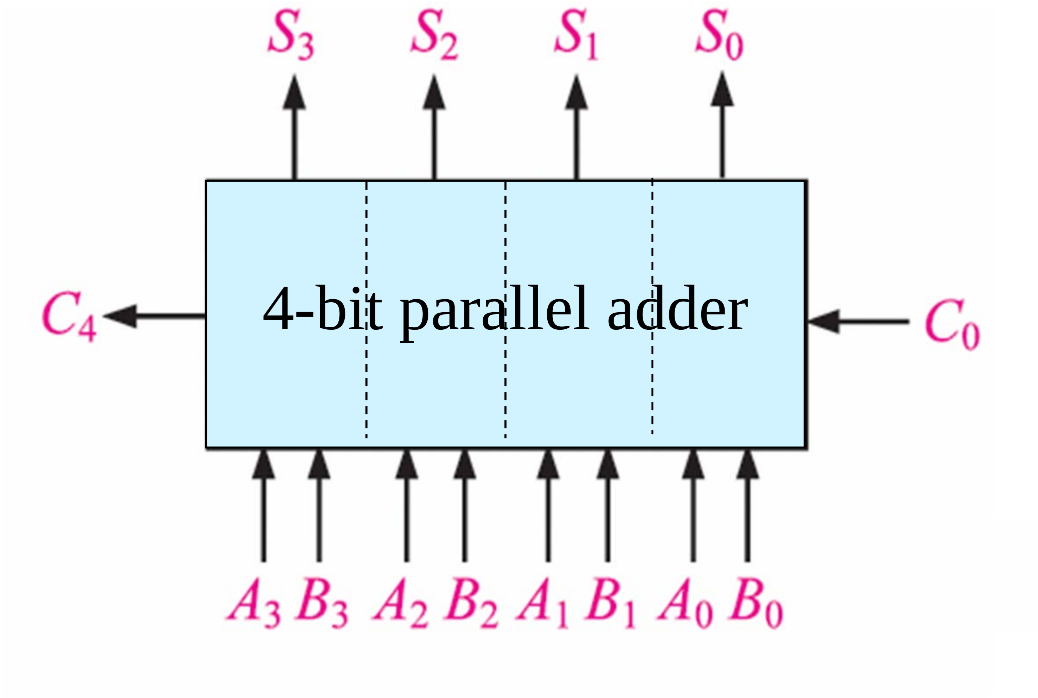
- **The logic circuit of full adder:**



Binary Adders & Subtracters (5/9)

- **4-Bit Parallel Adder (Ripple Carry Adder)**

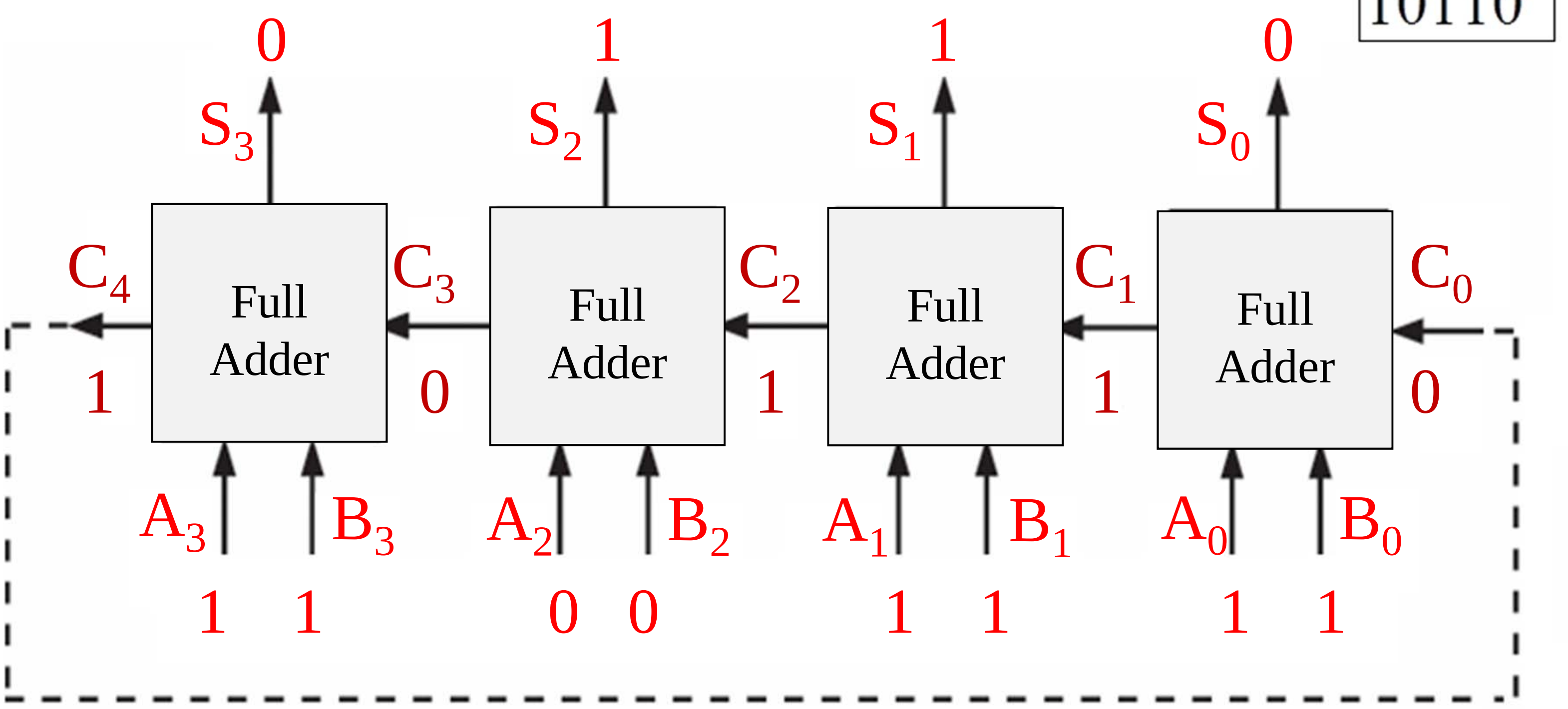
  Adds two 4-bit unsigned binary numbers



- **4-Bit Parallel Adder**

10110	(carries)
1011	
+1011	
10110	

Parallel Adder Composed of Four Full Adders

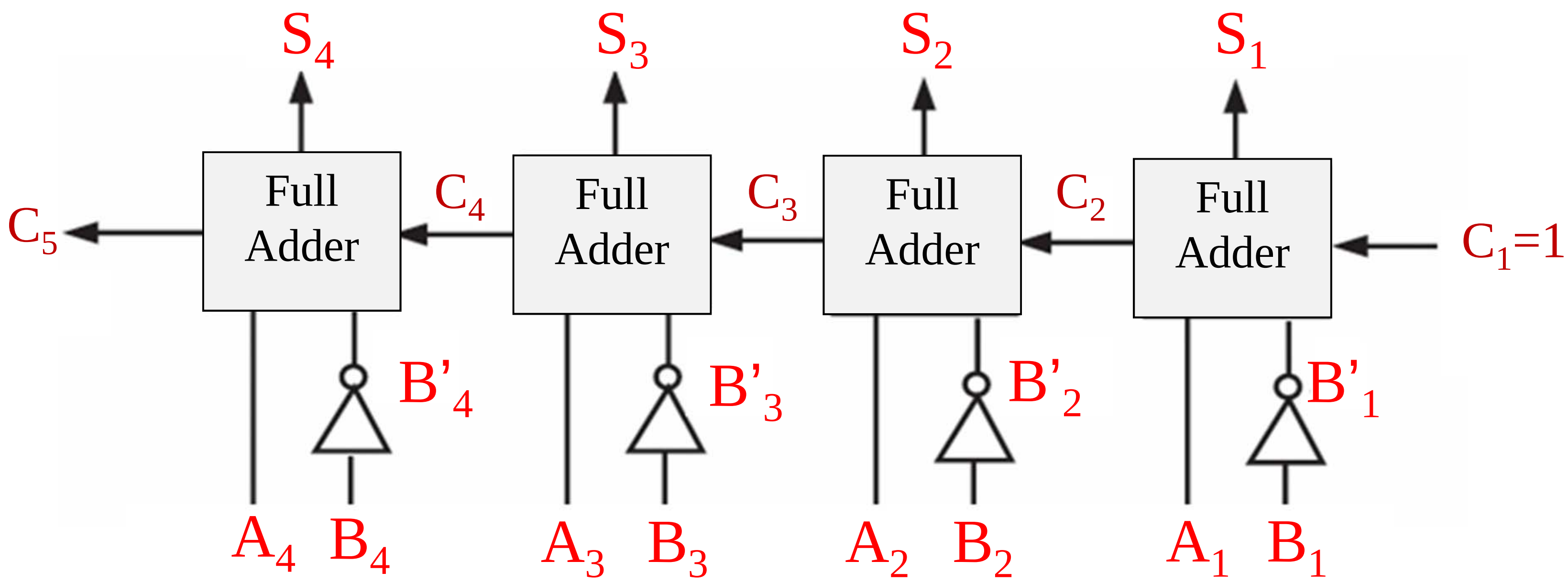


End-around carry for 1's complement

Binary Adders & Subtracters (7/9)

- Binary Subtracter Using Full Adders:

Subtraction of binary numbers is most easily accomplished by adding the complement of the number to be subtracted



Binary Subtracter Using Full Adder

Binary Adders & Subtracters (8/9)

- **Full Subtracter**

x_i	y_i	b_i	b_{i+1}	d_i
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	1	0
1	0	0	0	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

Truth Table for Binary Full subtracter

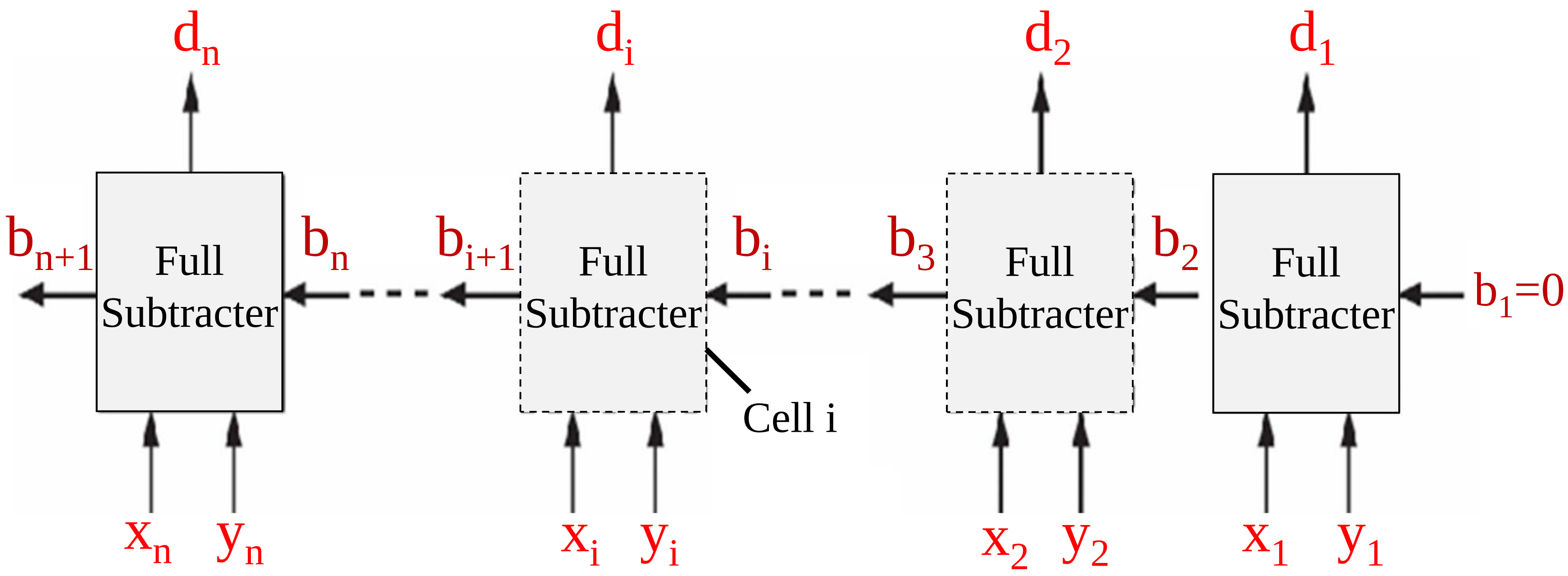
$x_i=0$, $y_i=1$, $b_i=1$

	Column i Before Borrow	*Column i After Borrow*
x_i	0	10
$-b_i$	-1	-1
$-y_i$	$\underline{-1}$	$\underline{-1}$
d_i		$0\ (b_{i+1}=1)$

Binary Adders & Subtracters (9/9)

- **Parallel Subtracter** :

  Direct subtraction can be accomplished by employing a subtracter.



Speeding Up Integer Additions

- Ripple carry adder
  - Simple, regular
  - Long delay
    - Last carry out needs 2n delay for n-bit adder

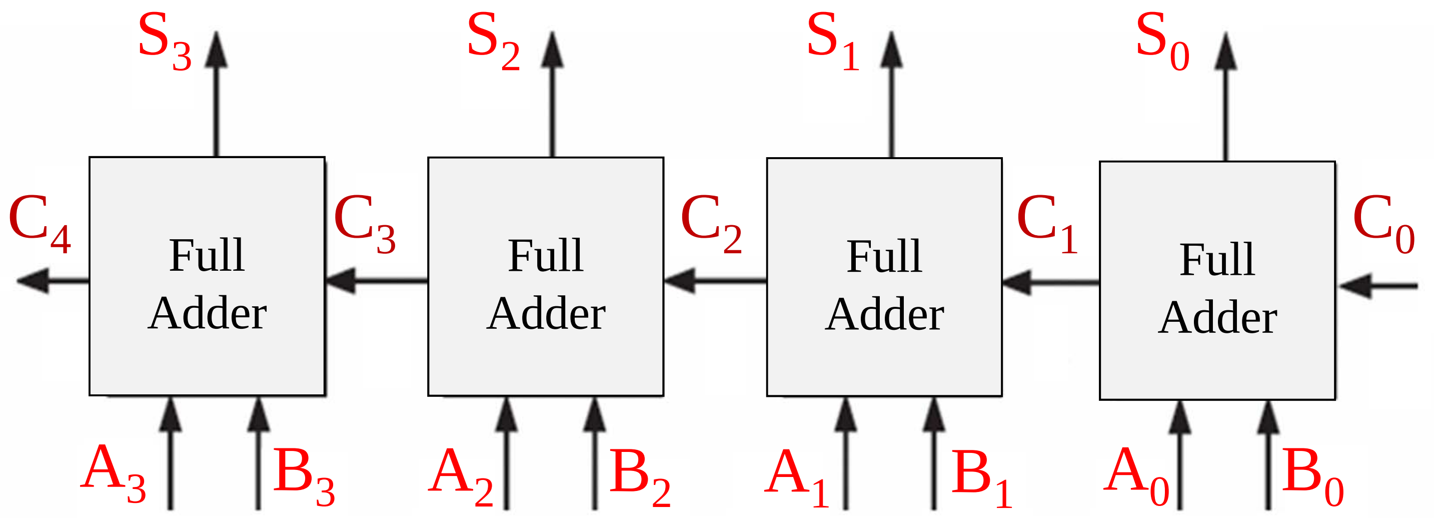
- Carry Lookahead Adder (CLA)

- Carry Select Adder

Carry Lookahead Adder (1/4)

$$Sum = A \oplus B \oplus C_{in}$$

$$C_{out} = AB + (A + B)C_{in}$$



$$C_i = AB + (A + B)C_{i-1}$$

$$C_i = A_{i-1}B_{i-1} + (A_{i-1} + B_{i-1})C_{i-1}$$

Carry Lookahead Adder (2/4)

$$C_i = A_{i-1}B_{i-1} + (A_{i-1} + B_{i-1})C_{i-1}$$

$g_{i-1} = A_{i-1}B_{i-1};$ generate function

$p_{i-1} = A_{i-1} + B_{i-1};$ propagate function

$$C_i = g_{i-1} + p_{i-1}C_{i-1}$$

$$C_1 = g_0 + p_0C_0$$

$$C_2 = g_1 + p_1C_1 = g_1 + p_1(g_0 + p_0C_0)$$
$$= g_1 + p_1g_0 + p_1p_0C_0$$

Carry Lookahead Adder (3/4)

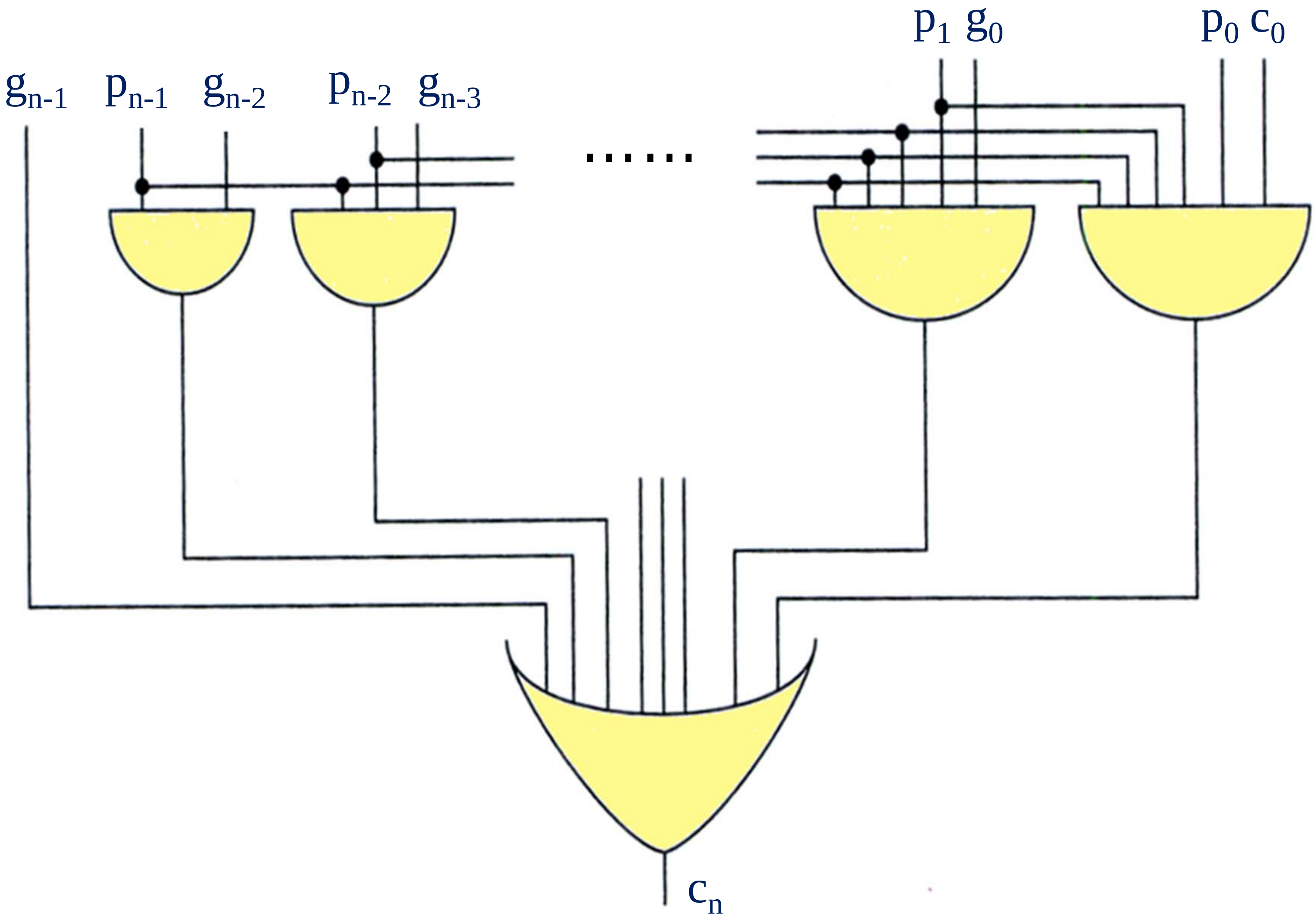
$$C_3 = g_2 + p_2 C_2 = g_2 + p_2(g_1 + p_1 g_0 + p_1 p_0 C_0)$$
$$= g_2 + p_2 g_1 + p_2 p_1 g_0 + p_2 p_1 p_0 C_0$$

$$C_n = g_{n-1} + p_{n-1} g_{n-2} + p_{n-1} p_{n-2} g_{n-3} + \cdots$$
$$+ p_{n-1} p_{n-2} \cdots p_1 g_0 + p_{n-1} p_{n-2} \cdots p_0 C_0$$

Delay of 4-bit adder: 2+2×3 =8 (ripple carry adder)

2+3=5 (CLA)

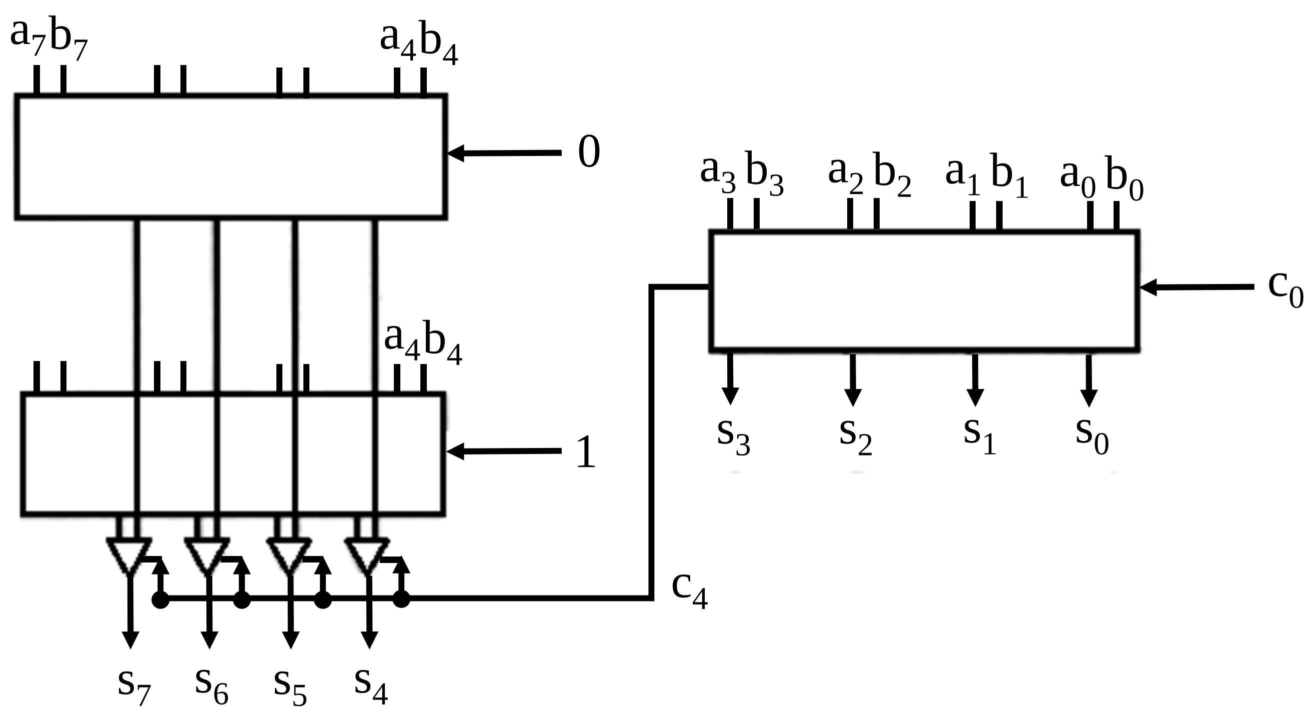
Carry Lookahead Adder (4/4)



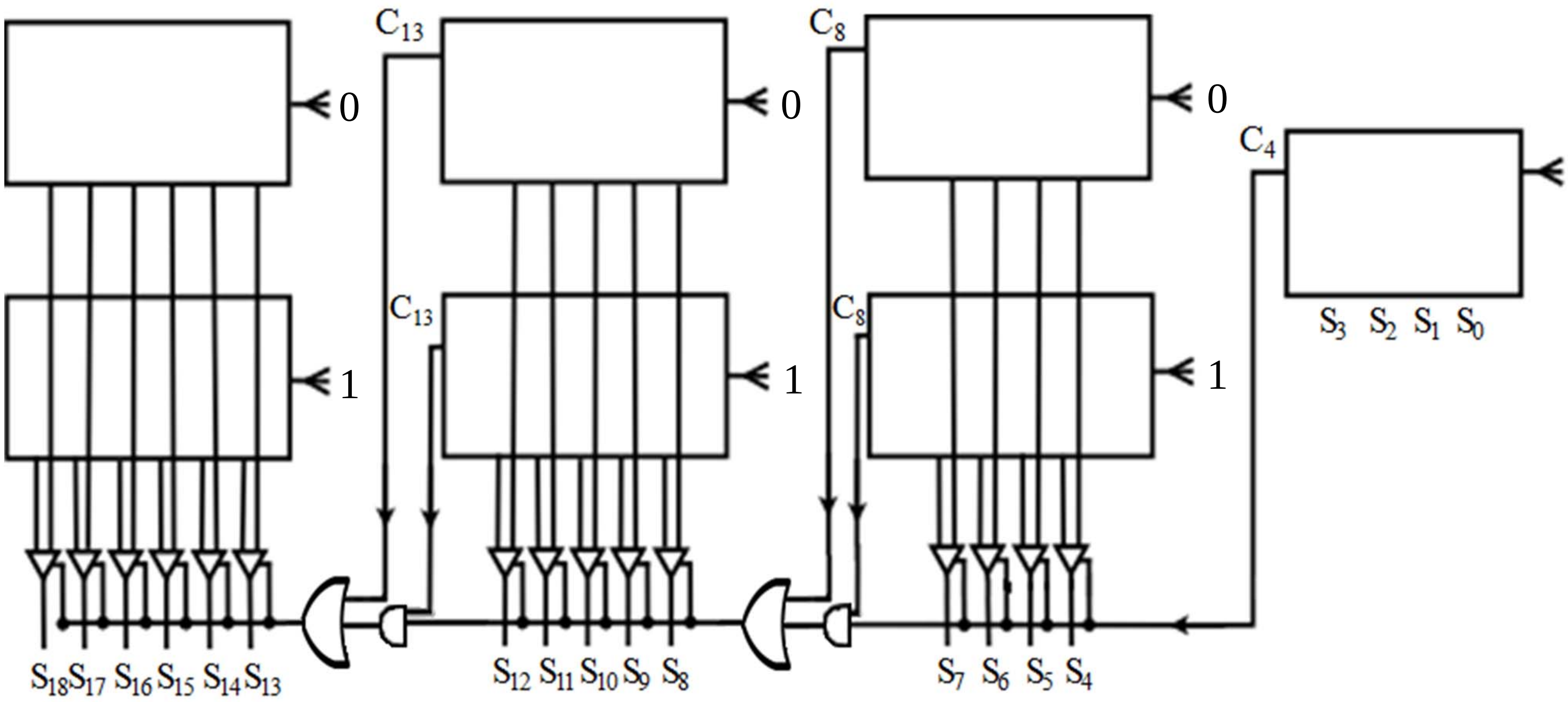
$$c_n = g_{n-1} + p_{n-1}g_{n-2} + \dots + p_{n-1}p_{n-2}\cdots p_1 g_0 + p_{n-1}p_{n-2}\cdots p_0 c_0$$

Carry Select Adder (1/2)

- Two additions are performed in parallel
  - One assumes the carry-in is 0; the other assumes 1
- When the carry-in is finally known, the correct sum is selected (has been precomputed)



Carry Select Adder (2/2)



Binary Multiplication

- Performed in the same way as with decimal numbers
  - Multiplicand B, multiplier A
  - Partial product
  - Shift one bit left
  - Sum of partial products B

$$
\begin{array}{cccc}
 & & B_1 & B_0 \\
 & & A_1 & A_0 \\
\hline
 & & A_0B_1 & A_0B_0 \\
 & A_1B_1 & A_1B_0 & \\
\hline
C_3 & C_2 & C_1 & C_0
\end{array}
$$